Comment on "Universality in sandpiles"

Mario De Menech*
Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, D–01187 Dresden, Germany
(Received 16 February 2004; published 13 August 2004)

The characterization of most of the scaling properties in sandpile models relies on numerical simulations, which allow us to collect a large number of avalanche events; in lack of an accepted theoretical framework, the estimate of the properties of probability distributions for an infinite system is based on empirical methods. Within the finite-size scaling hypothesis, for example, the scaling of the total energy dissipation s with the area a covered by the avalanche should follow the simple law $s \sim a^{\gamma_{sa}}$, with γ_{sa} marking the universality class of the model; γ_{sa} is normally measured from the scaling of the average value of s given a. Chessa *et al.* [Phys. Rev. E **59**, R12 (1999)] introduced a new procedure to extrapolate γ_{sa} for the Bak-Tang-Wiesenfeld model [P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988)], which leads to a value that matches the analogous exponent obtained for the Manna sandpile [S. S. Manna, J. Phys. A **24**, L363 (1991)], in support of the hypothesis of a unique universality class for the two models. This procedure is discussed in detail here; it is shown how the correction used by Chessa *et al.* depends on the lattice size L and disappears as $L \to \infty$.

DOI: 10.1103/PhysRevE.70.028101 PACS number(s): 05.65.+b, 05.40.−a

Sandpile models have been introduced as prototypes of self-organized criticality (SOC) [1,2], a conceptual framework which tries to explain the origin of self-similarity in natural systems where an external driving force causes sudden relaxations that occur with an intermittent pattern. Their popularity among theoreticians stems mainly from the simplicity of the dynamical rules, which suffice to produce a wealth of complex features analogous to those found in real systems. Nonetheless, the hope that some of these automata could acquire the status of Ising models for SOC has been frustrated over the years by conflicting results both at the theoretical and the numerical level, which has led to some confusion in the determination of universality classes. This comment addresses a technical aspect in the extrapolation of critical exponents from numerical simulations, hoping to contribute to the clarification of the general picture in the field.

The fluctuating driving force of a SOC system is represented in the sandpile by the random addition of grains to the nodes of a discrete lattice; grains can accumulate in a site until it becomes metastable, such that further addition causes the redistribution of grains to the neighboring sites, following a *toppling rule* that mimics the local nonlinear response in a SOC system. The toppling of a site may trigger the toppling of adjacent nodes, therefore propagating the instability over wider portions of the lattice. The sequence of topplings originated by a grain addition is identified as an avalanche event. Grains are dissipated by sink nodes, and the continuous input of grains drives the pile to a state characterized by a broad distribution of avalanche magnitudes, analogous to the wide fluctuation of activity bursts observed in SOC systems.

Numerical experiments have shown that avalanches in sandpile models have broadly distributed magnitudes, with a probability density function (PDF) that exhibits power-law behavior with cutoffs determined by the size of the system, suggesting the lack of a characteristic avalanche size in the thermodynamic limit. Being supported by very few exact results, scientist still have to resort to the statistical analysis of avalanches generated in computer simulations in order to extrapolate the exponents which rule the decay of PDF's as the linear size L of the system goes to infinity.

The Bak-Tang-Wiesenfeld (BTW) model is certainly the most studied among sandpile automata, often in comparison with its stochastic version, the Manna model [3]. The issue of the universality class of these two models has been long debated, and to date, most numerical results support the conjecture that they belong to different universality classes [4–6]. The paper by Chessa *et al.* [7] is one of those which still stands against this hypothesis, although some of its results have been already contrasted [6] on the grounds of a more asymptotic analysis of the moment scaling behavior. In this comment we address the "systematic bias," which, according to the authors, affects the data analysis in the BTW model.

Various quantities can be defined in order to assign a measure to an avalanche; the area a is the number of sites that topple at least once, giving the extension of the avalanche cluster, and the size s is defined as the total number of topplings occurred, representing the total energy dissipated by the avalanche. The signature of scaling can be observed by looking at the growth of the avalanche size with its area; in the finite-size scaling (FSS) hypothesis one would expect [5]

$$s \sim a^{\gamma_{sa}}, \tag{1}$$

where the exponent γ_{sa} determines the density of topplings in an avalanche. Since $s \geqslant a$, we have $\gamma_{sa} \geqslant 1$; in particular, $\gamma_{sa}=1$ means that size and area are equivalent measures, while $\gamma_{sa}>1$ implies the presence of several multiple topplings in the same avalanche. Equation (1) is tested by measuring the growth of the average size with the area, defined by $\langle s\rangle_a = \int ds\, s\, p(s|a)$, where $p(s|a)$ is the conditional probability. γ_{sa} is obtained from the linear fit of the plot $\log\langle s\rangle_a$

*Electronic address: mario@mpipks-dresden.mpg.de

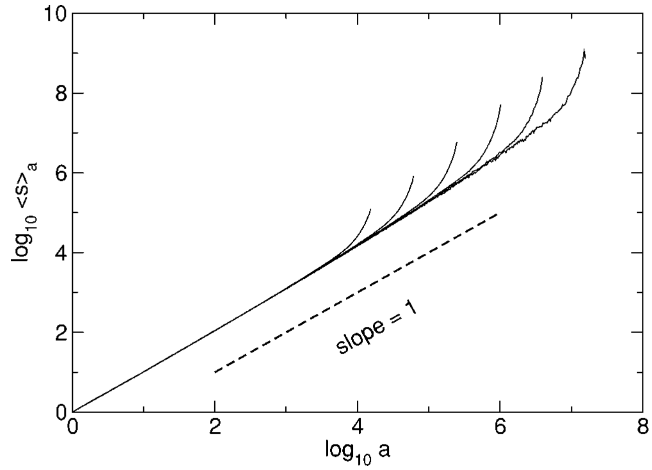

FIG. 1. The double logarithmic plots of the average size s as a function of the area a; data from $L = 128, 256, 512, 1024, 2048$, and 4096 are superimposed. Large avalanches having $a \approx L^2$ do not follow the simple scaling law $\langle s\rangle_a \sim a^{\gamma_{sa}}$, and are discarded in the determination of γ_{sa}.

vs log a. The "classical" value of the exponent determined in this way is $\gamma_{sa} = 1.06 \pm 0.01$ [4,8], neglecting the very large avalanches of area $a \approx L^2$, which have a much higher density of topplings and do not comply with the power-law behavior of Eq. (1) (see Fig. 1). It has been pointed out [4] that exponents related to conditional probabilities do not depend on system size, compared to distribution exponents, a fact which has given them a relevant role in the determination of universality classes.

Chessa *et al.* argue that for the BTW model, the numerical determination of γ_{sa}, as described above, is biased by finite-size effects, which, in their view, should be corrected by subtracting the area a to $\langle s\rangle_a$, in order to compensate for the asymmetry of $p(s|a)$. The plot giving γ_{sa} should be, therefore, $\log(\langle s\rangle_a - a)$ vs log a rather than $\log\langle s\rangle_a$ vs log a; Fig. 2 shows corrected plots for a range of sizes L. Apparently the correction leads to a significantly higher $\gamma_{sa} = 1.35 \pm 0.05$ [7],

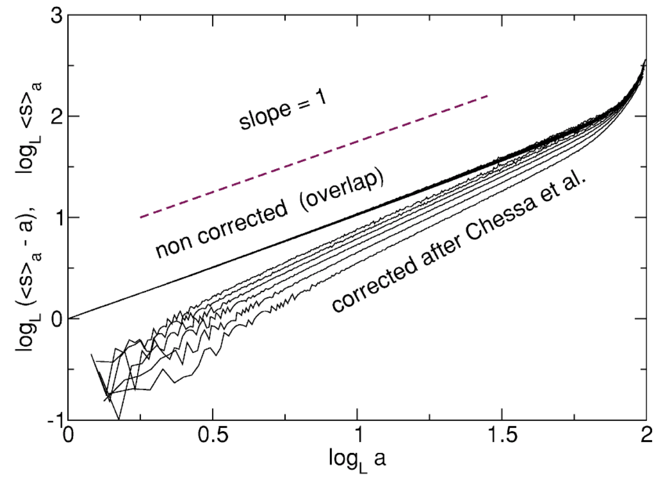

FIG. 3. The rescaled noncorrected and corrected plots from Figs. 1 and 2.

matching the analogous exponent obtained for the Manna model, and therefore supporting the hypothesis that the two belong to the same universality class.

A more accurate analysis shows that the two methods, as $L \to \infty$, give the same classical result, i.e., $\gamma_{sa} \simeq 1.06$. By rescaling data sets from different lattice sizes displayed in Figs. 1 and 2 by log L, one sees clearly that the slope of the corrected plot decreases with L, while the noncorrected curves overlap (Fig. 3) quite nicely. The γ_{sa} from the corrected plots at various lattice sizes is shown in Fig. 4: the intercept of the linear fit gives $\gamma_{sa}(L \to \infty) = 1.07 \pm 0.02$, confirming previous measurements. Such a simple check is quite effective in showing that the analysis of the scaling of the conditional probability described in Ref. [7] misses the L dependence of γ_{sa}.

The analysis described above agrees with the numerical results that suggest that the BTW and the Manna model be-

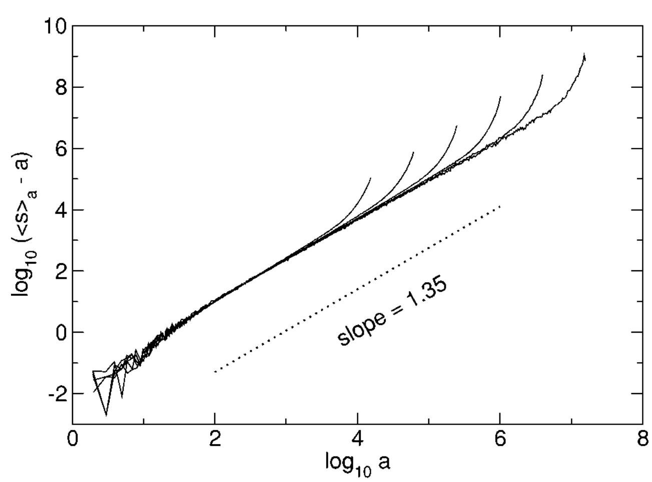

FIG. 2. The corrected plots for the scaling of the avalanche size with the avalanche area, according to Ref. [7] ($L = 128, 256, 512, 1024, 2048, 4096$).

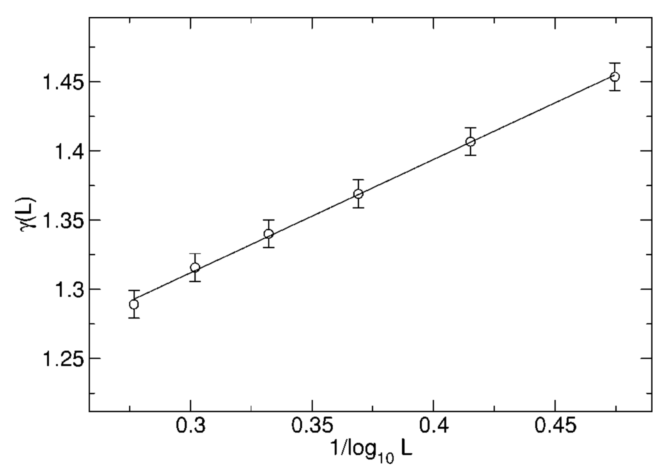

FIG. 4. The linear regression for the extrapolation of $\gamma_{sa}(\infty)$ from the slopes at different L values measured from the corrected plots in Fig. 3.

long to different universality classes. More generally, I would like to recall here that the differences between the two are more profound than expected [9,10]; while for the Manna model the exponent γ_{sa} fits nicely into a FSS framework, in the BTW case, one needs to define a distribution of exponents, as a result of the multiscaling character of the conditional probability The pecularity of the BTW model lies in the fact that the number of topplings in an avalanche of fixed area widely fluctuates even as we increase L, both for dissipating and nondissipating avalanches.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[2] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
[3] S. S. Manna, J. Phys. A **24**, L363 (1991).
[4] A. Ben-Hur and O. Biham, Phys. Rev. E **53**, R1317 (1996).
[5] K. Christensen, H. C. Fogedby, and H. J. Jensen, J. Stat. Phys. **63**, 653 (1991).
[6] S. Lübeck, Phys. Rev. E **61**, 204 (2000).
[7] A. Chessa, H. E. Stanley, A. Vespignani, and S. Zapperi, Phys. Rev. E **59**, R12 (1999).
[8] K. Christensen and Z. Olami, Phys. Rev. E **48**, 3361 (1993).
[9] C. Tebaldi, M. D. Menech, and A. L. Stella, Phys. Rev. Lett. **83**, 3952 (1999).
[10] M. D. Menech, A. L. Stella, and C. Tebaldi, Phys. Rev. E **58**, R2677 (1998).